

# REPRESENTATIONS WITH $Sp(1)^k$ -REDUCTIONS AND QUATERNION-KÄHLER SYMMETRIC SPACES

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**ABSTRACT.** We classify non-polar irreducible representations of connected compact Lie groups whose orbit space is isometric to that of a representation of a finite extension of  $Sp(1)^k$  for some  $k > 0$ . It follows that they are obtained from isotropy representations of certain quaternion-Kähler symmetric spaces by restricting to the “non- $Sp(1)$ -factor”.

## 1. INTRODUCTION

The aim of this paper is to contribute to the program initiated in [GL14], namely, hierarchize the representations of compact Lie groups in terms of the complexity of their orbit spaces, viewed as metric spaces. We say that two representations are *quotient-equivalent* if they have isometric orbit spaces. Given a representation, if there is a quotient-equivalent representation of a lower-dimensional group, then we say that the former representation *reduces* to the latter one and that the latter representation is a *reduction* of the former one. A *minimal reduction* of a representation is a reduction with smallest possible dimension of the underlying group.

At the basis of the hierarchy lie the *polar representations*, namely, those representations that reduce to finite group actions, and which turn out to be related to symmetric spaces (indeed every polar representation of a connected compact Lie group has the same orbits as the isotropy representation of a symmetric space [Dad85]). In [GL15], there were studied and classified those irreducible representations of connected groups that reduce to actions of groups with identity component a torus  $S^1 \times \cdots \times S^1$ . It was shown that, mostly, those are close relatives of Hermitian symmetric spaces. Herein we study the quaternionic version, namely, those irreducible representations of connected groups that reduce to an action of a group whose identity component is a “quaternionic torus”  $S^3 \times \cdots \times S^3$ . Interestingly enough, these are related to quaternion-Kähler symmetric spaces, in a stricter sense than in the Hermitian case.

**Theorem.** *Let  $\tau : H \rightarrow O(W)$  be a non-polar irreducible representation of a connected compact Lie group. Assume that  $\tau$  is quotient-equivalent to a representation  $\rho : G \rightarrow O(V)$  where  $G^0 \cong Sp(1)^k$  for some  $k > 0$  and  $\dim G < \dim H$ . Then  $k = 3$ ,  $G$  is disconnected and  $V = \otimes^3 \mathbb{C}^2$ ; moreover, the cohomogeneity of  $\tau$  is 7 and it is*

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obtained by restricting the isotropy representation of a certain quaternion-Kähler symmetric space to the “non- $\mathrm{Sp}(1)$ -factor”. More precisely,  $\tau$  is one of:

| $\tau$   | $qK$ symmetric space                                      | $G/G^0$ |
|--|---|---------|
| $(\mathrm{SO}(n) \times \mathrm{Sp}(1), \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{H})$ | $\mathrm{SO}(n+4)/(\mathrm{SO}(n) \times \mathrm{SO}(4))$ | $A_1$   |
| $(\mathrm{Sp}(3), \Lambda^3 \mathbb{C}^6 \ominus \mathbb{C}^6)$                        | $F_4/(\mathrm{Sp}(3)\mathrm{Sp}(1))$                      | $A_2$   |
| $(\mathrm{SU}(6), \Lambda^3 \mathbb{C}^6)$   | $E_6/(\mathrm{SU}(6)\mathrm{SU}(2))$                      | $A_2$   |
| $(\mathrm{Spin}(12), \mathbb{C}^{32})$   | $E_7/(\mathrm{Spin}(12)\mathrm{SU}(2))$                   | $A_2$   |
| $(E_7, \mathbb{C}^{56})$   | $E_8/(E_7\mathrm{SU}(2))$                                 | $A_2$   |

Finally,  $\rho$  is a minimal reduction of  $\tau$ .

Note that there are three additional families of quaternion-Kähler symmetric spaces absent from the Table, namely, the restrictions of their isotropy representations do not have the kind of reduction as in the Theorem. The Theorem says in particular that  $k$  must be odd, so  $\rho$  is a representation of quaternionic type. A posteriori we find that the  $\mathrm{Sp}(1)$ -group of isometries of  $W/H = V/G$  induced by the normalizer of  $G$  in  $\mathrm{O}(V)$  can be lifted to a  $\mathrm{Sp}(1)$ -subgroup of  $\mathrm{O}(W)$ . In this connection, we note that the general problem of lifting isometries of orbit spaces to isometries deserves further attention ([AL11, Question 1.6] and [GL14, Question 1.13]).

The structure of the proof of the Theorem goes as follows. The situation of a non-trivial reduction from  $\tau$  to  $\rho$  entails the presence of non-empty boundary for the orbit space of  $\rho$ , according to results in [GL14, §5]. A careful analysis of the existence of boundary points then forces  $k = 3$ . Now  $\rho|_{G^0}$  is of quaternionic type, so its orbit space admits an  $\mathrm{Sp}(1)$ -group of isometries. An application of a theorem of Thorbergsson, as in [GL15], shows that  $\tau$  must be the restriction of a polar representation of Coxeter type  $B_4$  or  $F_4$ . The argument is finished by invoking Dynkin’s classification of maximal connected closed subgroups of compact Lie groups.

We follow notation and terminology from [GL14]. See also [GL16] for some background material on stratification of orbit spaces and their metric structures. The authors wish to thank Alexander Lytchak for several valuable comments.

## 2. STRUCTURE OF THE EXAMPLES

In this section we show that the representations  $\tau$  listed in the table of the Theorem indeed admit reductions to a group whose identity component is  $\mathrm{Sp}(1)^3$ . In all cases, the reduction in hand is the Luna-Richardson-Straume reduction to the normalizer of a principal isotropy group acting on the fixed point set of this principal isotropy group [GL16, §2.6]. It will follow from the discussion in Subsection 3.3 that those reductions are minimal.

**2.1. The real Grassmannian.** The isotropy representation of the rank 4 real Grassmannian manifold is  $\mathrm{SO}(n) \times \mathrm{SO}(4)$  acting on  $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^4$ . In this case  $\tau$  is given by  $H = \mathrm{SO}(n) \times \mathrm{SU}(2)$  acting on  $\mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{C}^2$  with principal isotropy group  $H_{\mathrm{princ}} = \mathrm{SO}(n-4)$ , and its effective normalizer  $N_H(H_{\mathrm{princ}})/H_{\mathrm{princ}} = \mathrm{S}(\mathrm{O}(4) \times \mathrm{O}(n-4)) \times \mathrm{SU}(2)/H_{\mathrm{princ}} = \mathrm{O}(4) \times \mathrm{SU}(2) = \mathbb{Z}_2 \cdot \mathrm{Sp}(1)^3$  has the desired form.

**2.2. The exceptional cases.** For each one of the rank 4 exceptional quaternion-Kähler symmetric spaces, the isotropy representation  $\hat{\tau}$  is given by  $\hat{H} = H \cdot \mathrm{Sp}(1)$  acting on  $\hat{W} = W \otimes_{\mathbb{H}} \mathbb{H} \cong W$  and  $\tau = \hat{\tau}|_H : H \rightarrow \mathrm{O}(W)$  is a representation of quaternionic type. The principal isotropy group of  $\hat{\tau}$  has the form  $\hat{H}_{\mathrm{princ}} = H_{\mathrm{princ}} \cdot Q$  where  $Q$  is the quaternion group  $\{\pm 1, \pm i, \pm j, \pm k\}$  diagonally embedded in  $H \cdot \mathrm{Sp}(1)$ . Since  $\hat{\tau}$  is asystatic (see [Str94, pp.11-12] or [GK16, §2.2]), it is polar and the fixed point set  $W^{\hat{H}_{\mathrm{princ}}}$  is a section; this is a totally real subspace of  $W$  of dimension 4. Now  $W^{H_{\mathrm{princ}}}$  is a quaternionic subspace of  $W$ , and it must be the quaternionic span of  $W^{\hat{H}_{\mathrm{princ}}}$ , of real dimension 16. Since the cohomogeneity of  $\tau$  is  $c(\tau) = c(\hat{\tau}) + \dim \mathrm{Sp}(1) = 4 + 3 = 7$  (compare [HH70, Table A]), we deduce that  $\dim N_H(H_{\mathrm{princ}})/H_{\mathrm{princ}} = \dim W^{H_{\mathrm{princ}}} - c(\tau) = 16 - 7 = 9$ . The group  $\dim N_H(H_{\mathrm{princ}})/H_{\mathrm{princ}}$  acts irreducibly on  $W^{H_{\mathrm{princ}}}$ , so its center is at most one-dimensional. A quick enumeration of the possible groups reveals that  $\dim N_H(H_{\mathrm{princ}})/H_{\mathrm{princ}}$  is locally isomorphic to  $\mathrm{Sp}(1)^3$  or  $\mathrm{U}(3)$ , but  $W^{H_{\mathrm{princ}}}$  is a representation of quaternionic type, and the latter group admits none. Note also that the only 16-dimensional irreducible representation of  $\mathrm{Sp}(1)^3$  is  $\otimes^3 \mathbb{C}^2$ .

### 3. PROOF OF THE THEOREM

Throughout this section we let  $\tau : H \rightarrow \mathrm{O}(W)$  and  $\rho : G \rightarrow \mathrm{O}(V)$  be as in the statement of the Theorem.

**3.1. Triviality of the principal isotropy group.** We observe that the principal isotropy group of  $\rho^0 := \rho|_{G^0}$  is trivial. Indeed we have the following general result.

**Lemma 1.** *Every irreducible representation of  $\mathrm{Sp}(1)^k$  is either polar or has trivial principal isotropy group.*

*Proof.* We shall collect the irreducible representations with possibly non-trivial principal isotropy group and check that they are all polar. The complexification of the principal isotropy group of a representation of  $\mathrm{Sp}(1)^k$  on  $V$  is the stabilizer in general position (sgp) of the corresponding complex representation of  $\mathrm{SL}(2, \mathbb{C})^k$  on  $V^c = V \otimes_{\mathbb{R}} \mathbb{C}$  [Sch80, §5]. If  $V$  has no invariant complex structure, then  $V^c$  is irreducible; otherwise,  $V^c$  consists of two copies of  $V$ . Now it suffices to start listing irreducible representations  $U$  of  $\mathrm{SL}(2, \mathbb{C})$  such that:

- $U$  has non-trivial sgp, if  $U$  is a representation of real type;
- $U \oplus U$  has non-trivial sgp, if  $U$  is a representation of quaternionic type;

and then checking that these admit an invariant real form  $V$  which is a polar representation of  $\mathrm{Sp}(1)^k$ .

At this point, one could apply [AVÉ67, Theorem] and refer to [Pop75, Table 1] and [Pop78, Table 1] to finish the job. Alternatively, the case  $k = 1$  is well known (see e.g. [PV94, p. 231]; here  $V = \mathbb{R}^3$  or  $\mathbb{R}^5$  which are polar). For  $k \geq 2$ , we work from scratch using [AP71, Theorem 2] (see also [PV94, Theorem 7.10]) to deduce that a necessary condition for non-triviality of sgp of  $V^c$  is that  $\dim V = \dim_{\mathbb{C}} V^c \leq 8k$ ; this estimate takes care of almost all representations. Additional representations are excluded with the help of the Corollary to Lemma 6 in [AP71]. We end up with few non-polar irreducible representations of  $\mathrm{Sp}(1)^k$ , each of which is easily checked to have trivial principal isotropy group.  $\square$

**3.2. General setting.** Note that  $V$  equals  $\mathbb{C}^{n_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}^{n_k}$  or a real form  $[\mathbb{C}^{n_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathbb{C}^{n_k}]_{\mathbb{R}}$  according to whether the dimension  $n_i$  is even for an odd, resp. even, number of indices  $i$ . We have that  $V$  is of quaternionic type in the first case, and of real type in the second one. We may assume  $2 \leq n_1 \leq \cdots \leq n_k$ .

Due to [GL14, §5],  $X = V/G = W/H$  has non-empty boundary as an Alexandrov space. We recall that the boundary is given by the closure of the codimension one strata of  $X$ . Since the principal isotropy is trivial, a point  $p \in V$  projects to such a stratum — called a *G-important point* in [GL14] — if and only if its isotropy group  $G_p$  is a sphere  $S^\ell$  of dimension  $\ell$  equal to 0, 1 or 3 and

$$(1) \quad \dim V - 1 - \ell = 3k - m + f,$$

where  $m$  is the dimension of the normalizer  $N_G(G_p)$  and  $f$  is the dimension of the fixed point set  $V^{G_p}$  of  $G_p$  in  $V$ , see [GL14, Lemma 4.1].

In the following two subsections, we will analyze the boundary of  $V/G$  and show that it can be non-empty only if  $G$  is disconnected,  $k = 3$  and  $V = \otimes^3 \mathbb{C}^2$ . The cases  $k = 1$  and  $k = 2$  were discussed in [GL14, §10], so we may assume  $k \geq 3$ .

**3.3. Connected case.** We first suppose  $G$  is connected and show that this assumption is incompatible with  $V/G$  having non-empty boundary. So suppose that  $p \in V$  is a  $G$ -important point. Since  $G$  is connected, no  $G$ -important point may lie in an exceptional orbit [Lyt10], so  $G_p$  is not discrete. Moreover, any  $\mathrm{SU}(2)$ -subgroup of  $G$  contains a unique involution that is central in  $G$ . Since such an involution cannot have fixed points by irreducibility, we cannot have  $G_p \cong \mathrm{SU}(2)$ . We deduce that  $G_p \cong S^1$ .

The dimension formula (1) yields

$$\theta \cdot n_1 \cdots n_k - 2 = 3k - m + f,$$

where  $\theta = 1$  or  $2$  in case  $V$  is of real or quaternionic type, respectively. It is moreover clear that  $m \geq k$ . Since  $V^{G_p}$ , resp. its complexification, is the sum of weight spaces whose weights lie in a hyperplane of the dual Lie algebra of the maximal torus of  $G$ , it is not hard to see that  $f \leq \theta \cdot n_2 \cdots n_k$ . We deduce that

$$\theta(n_1 - 1)n_2 \cdots n_k \leq 2k + 2.$$

In particular,  $\theta \cdot 2^{k-2} \leq k+1$  implying  $k = 3$  or  $4$ . If  $k = 3$ , we obtain  $(\mathrm{Sp}(1)^3, \otimes^3 \mathbb{C}^2)$  and  $(\mathrm{SO}(4) \times \mathrm{SO}(3), \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^3)$ ; the latter representation is polar, which cannot be. If  $k = 4$ , we obtain  $(\mathrm{SO}(4) \times \mathrm{SO}(4), \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^4)$  which is also polar and again is excluded.

It remains to analyze the case of  $(\mathrm{Sp}(1)^3, \otimes^3 \mathbb{C}^2)$ . Here the dimension formula (1) says that  $5 + m = f$ , and  $m \geq 3$  implying  $f \geq 8$ . Recall that the weights of  $\mathbb{C}^2$  are  $\pm \epsilon$ , where  $2\epsilon$  is the positive root of  $\mathrm{Sp}(1)$ . It is apparent that  $V^{G_p}$  can contain at most four weight spaces so  $f = 8$ . However, in this case  $G_p$  is a circle diagonally embedded in two factors of  $\mathrm{Sp}(1)^3$  which gives  $m = 5$  and contradicts the dimension formula. We deduce that  $G$  cannot be connected, as desired.

We have shown that the orbit space of any non-polar representation of  $\mathrm{Sp}(1)^k$  has empty boundary. Together with [GL14, Proposition 5.2], this implies that *an irreducible action of any extension of  $\mathrm{Sp}(1)^k$  by a finite group is either polar or reduced (i.e. it cannot be further reduced).*

**3.4. Disconnected case.** We now suppose  $G$  is disconnected and prove that  $V/G$  can have non-empty boundary only if  $k = 3$  and  $V = \otimes^3 \mathbb{C}^2$ . Since  $H$  is connected, there is an involution  $w \in G \setminus G^0$ , called a *nice involution*, that acts as a reflection on  $V/G^0$  (see Proposition 3.2 and §4.3 in [GL14]). The dimension formula reads

$$(2) \quad \dim V - 1 = \dim G - \dim C(w) + \dim V^w$$

where  $C(w)$  is the centralizer of  $w$  in  $G$  and  $V^w$  is the fixed point set of  $w$  in  $V$ . The element  $w$  acts on  $G^0$  by conjugation.

**3.4.1. Outer automorphism.** We first discuss the case in which  $w$  acts on  $G^0$  by an outer automorphism. Since  $w$  is not inner, its action on  $G^0$  induces a non-trivial involutive permutation  $\sigma$  of the factors, and its action on  $V$  induces a corresponding permutation of the factors of  $V$ . Consider  $w_0 \in \mathcal{O}(V)$  given by

$$w_0(v_1 \otimes \cdots \otimes v_k) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Then  $w_0^{-1}w$  induces an inner automorphism of  $G^0$ , so we can write  $w = w_0hz$  where  $h = (h_1, \dots, h_k) \in G^0$  and  $z \in \mathcal{O}(V)$  centralizes  $G^0$ .

Consider  $V$  as a representation  $\rho^0$  of  $G^0$  and relabel the indices to write

$$V = V_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V_a \otimes_{\mathbb{R}} \mathbb{R}^{n_1} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathbb{R}^{n_b} \otimes_{\mathbb{R}} \mathbb{H}^{p_1/2} \otimes_{\mathbb{H}} \cdots \otimes_{\mathbb{F}} \mathbb{H}^{p_c/2}$$

where  $n_i \geq 3$  is odd,  $p_j \geq 2$  is even,  $\mathbb{F} = \mathbb{H}$  or  $\mathbb{R}$  according to whether  $c$  is even or odd; moreover,  $V_i = \mathbb{R}^{m_i} \otimes_{\mathbb{R}} \mathbb{R}^{m_i}$  or  $V_i = \mathbb{H}^{m_i/2} \otimes_{\mathbb{R}} \mathbb{H}^{m_i/2}$  according to whether  $m_i$  is odd or even, and  $w_0$  exchanges the factors of  $V_i$  for  $i = 1, \dots, a$ , and fixes the other factors of  $V$ . If  $c$  is even, then  $V$  is of real type, so  $z = \pm 1$ ; if  $c$  is odd, then  $V$  is of quaternionic type, so  $z$  is right multiplication on  $\mathbb{H}^{p_c/2}$  by some element of  $\mathrm{Sp}(1)$  (we view quaternionic vector spaces as right  $\mathbb{H}$ -modules); in the latter case,  $w_0$  fixes the factor  $\mathbb{H}^{p_c/2}$ , so in any case  $w_0$  commutes with  $z$ . Now  $w^2 = 1$  gives that

$$(h_{\sigma(1)}h_1, \dots, h_{\sigma(k)}h_k) = z^{-2} \in Z(G^0) = \{\pm 1\}^k.$$

We deduce that  $h_{2i} = \pm h_{2i-1}^{-1}$  for  $i = 1, \dots, a$ . Now we can take

$$\tilde{h} = (1, h_1^{-1}, \dots, 1, h_\ell, 1, \dots, 1) \in G^0,$$

where  $\ell = 2a - 1$ , to replace  $w$  by the conjugate element  $\tilde{w} = \tilde{h}w\tilde{h}^{-1}$  in somehow simpler form, namely,

$$\tilde{w} = w_0z(1, \pm 1, \dots, 1, \pm 1, h_{\ell+1}, \dots, h_k)$$

(we could also have some simplification for the  $h_i$  for  $i > \ell$ , but it is unimportant to the sequel).

We have

$$\dim V = m_1^2 \cdots m_a^2 \cdot n_1 \cdots n_b \cdot p_1 \cdots p_c \cdot \theta,$$

where  $\theta = 1$  or  $2$  whether  $c$  is even or odd,

$$\dim G = 3(2a + b + c) \quad \text{and} \quad \dim C(w) = \dim C(\tilde{w}) \geq 3a + b + c.$$

The  $\pm 1$ -eigenspaces of  $w_0$  on  $V_i$  have dimension  $\frac{m_i(m_i \pm 1)}{2} =: m_i^\pm$ . We deduce that

$$\dim V^w = \dim V^{\tilde{w}} \leq \dim V^{\pm w_0} \leq \dim V^{w_0} = M \cdot n_1 \cdots n_b \cdot p_1 \cdots p_c \cdot \theta$$

where

$$M^{even} = \sum_5 m_1^\pm \cdots m_a^\pm$$

and the sum runs through all combinations with an even number of negative signs. The dimension formula (2) now gives

$$m_1 \cdots m_a n_1 \cdots n_b p_1 \cdots p_c \theta (m_1 \cdots m_a - M^{even}) \leq 3a + 2b + 2c + 1.$$

The factor between parenthesis is  $M^{odd} = \sum m_1^\pm \cdots m_a^\pm$  where the combinations are now taken with an odd number of negative signs. Since  $m_i^+ \geq \frac{3}{2}$  and  $m_i^- \geq \frac{1}{2}$ , we estimate

$$\begin{aligned} M^{odd} &= \sum_{\ell=0}^{\lfloor \frac{a-1}{2} \rfloor} \binom{a}{2\ell+1} \frac{3^{a-(2\ell+1)}}{2^a} \\ &= \left(\frac{3}{2}\right)^a \frac{1}{2} \left[ \left(1 + \frac{1}{3}\right)^a - \left(1 - \frac{1}{3}\right)^a \right] \\ &= \frac{1}{2} (2^a - 1) \\ &\geq 2^{a-2}. \end{aligned}$$

We deduce that

$$2^{2a-2} 3^b 2^{c+1} \cdot \theta \leq 3a + 2b + 2c + 1.$$

It immediately follows that  $a \leq 2$ , and a run through the possibilities, excluding polar representations, yields that  $\rho^0$  must be  $(\mathbf{Sp}(1)^3, \otimes^3 \mathbb{C}^2)$ .

**3.4.2. Inner automorphism.** We next consider the case in which  $w$  acts on  $G^0$  as an inner automorphism and show that this case gives nothing. Write  $w = qj$  where  $q$  centralizes  $G^0$  and  $j \in G^0$ . Write also

$$V = \mathbb{R}^{m_1} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathbb{R}^{m_a} \otimes_{\mathbb{R}} \mathbb{H}^{n_1/2} \otimes_{\mathbb{H}} \cdots \otimes_{\mathbb{F}} \mathbb{H}^{n_b/2}$$

where  $m_i \geq 3$  is odd and  $n_j \geq 2$  is even. Suppose  $V$  is of real type ( $b$  is even). Then  $q = \pm 1$ . Since  $q$  does not lie in  $G^0$ , we must have  $q = -1$  and  $b = 0$ , namely, all factors of  $G^0$  are isomorphic to  $\mathbf{SO}(3)$  and  $w = -j$ , where  $j \in G^0$ ,  $j^2 = 1$ . Write  $j = j_1 \cdots j_a$  where  $j_i$  is the component of  $j$  in the  $i$ th factor of  $G^0$ , and assume  $j_i \neq 1$  precisely for  $1 \leq i \leq a'$  for some  $0 \leq a' \leq a$ . On one hand, the dimension formula (2) gives

$$\begin{aligned} \dim V^j &= \dim V - \dim V^w \\ &= \dim G - \dim C(w) + 1 \\ &= 3a - (a' + 3(a - a')) + 1 \\ &= 2a' + 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} \dim V^j &= m_{a'+1} \cdots m_a \dim V^{j_1 \cdots j_{a'}} \\ &= m_{a'+1} \cdots m_a \sum g_1 \cdots g_{a'} \end{aligned}$$

where  $g_i = e_i$  or  $f_i$  and the sum runs through all possibilities with an even number of  $f_i$ 's; here

$$e_i = \dim V^{j_i} = \begin{cases} p_i & \text{if } p_i \text{ is odd,} \\ p_i + 1 & \text{if } p_i + 1 \text{ is even} \end{cases}$$

and

$$f_i = \dim V^{-j_i} = \begin{cases} p_i & \text{if } p_i + 1 \text{ is odd,} \\ p_i + 1 & \text{if } p_i \text{ is even} \end{cases}$$

and  $m_i = 2p_i + 1 \geq 3$  for all  $i$ .

Since  $e_i \geq 1$  and  $f_i \geq 2$ , we estimate

$$\begin{aligned} \sum g_1 \cdots g_{a'} &= e_1 \cdots e_{a'} + f_1 f_2 e_3 \cdots e_{a'} + \cdots \\ &\geq \sum_{\ell=0}^{\lfloor \frac{a'}{2} \rfloor} \binom{a'}{2\ell} 2^{2\ell} \\ &= \frac{1}{2} [3^{a'} + (-1)^{a'}]. \end{aligned}$$

We deduce that

$$2a' + 1 \geq m_{a'+1} \cdots m_a \frac{1}{2} [3^{a'} + (-1)^{a'}] \quad \text{and} \quad 2a' + 1 \geq \frac{1}{2} [3^{a'} + (-1)^{a'}].$$

The second inequality can be satisfied only if  $a' \leq 2$ ; in this case, using  $a \geq 3$ , we see that the first inequality is never satisfied. Hence  $V$  cannot be of real type.

We finally take up the case  $V$  is of quaternionic type ( $b$  is odd). Then  $q^2 = j^{-2}$  lies in the center of  $G^0$ , which is isomorphic to  $(\mathbb{Z}_2)^b$ , and  $q$  is a unit quaternion multiplying on the right. It follows that  $q^2 = \pm 1$ . If  $q^2 = 1$ , then irreducibility of  $G^0$  implies that one of the  $\pm 1$ -eigenspaces of  $q$  is trivial, namely,  $q = \pm 1$ , but then  $q \in G^0$  as  $b$  is odd, a contradiction. If  $q^2 = -1$ , then  $q$  defines a complex structure with respect to which  $w$  is a complex involution. In particular, the fixed point set  $V^w$  has even dimension. Note that  $m := \dim C(w) \in \{a + b, a + b + 2, \dots, a + 3b\}$  implying  $m \equiv a + b \pmod{2}$ . The dimension formula yields

$$\dim V - 1 = 3(a + b) - m + \dim V^w$$

leading to a contradiction as  $\dim V$  is even, too.  $\square$

**3.5. Discrete part.** It follows from the discussion in the previous two subsections that  $G \neq G^0 = \mathrm{Sp}(1)^3$  and  $V = \otimes^3 \mathbb{C}^2$ . The next step is to determine the exact nature of  $G$ .

**Lemma 2.** *The group  $G = \Gamma \ltimes G^0$  where  $\Gamma = A_1$  or  $A_2$ .*

*Proof.* Recall that the Coxeter group  $A_n$  is the group of permutations on  $n + 1$  elements. We saw in Subsection 3.4 that there exists a nice involution  $w \in G \setminus G^0$  acting on  $G^0$  by an outer automorphism. Note that  $\dim V = 16$ ,  $\dim G = 9$  and  $m = 6$  since  $\sigma$  permutes two factors of  $G^0$ . The dimension formula implies that  $f = 12$  and hence  $w$  operates on  $\otimes^3 \mathbb{C}^2$  by interchanging two factors. Since  $G$  acts with trivial principal isotropy groups,  $G/G^0 \rightarrow \mathrm{Isom}(V/G^0)$  is injective; moreover, its image is a group generated by reflections. We select a generating set consisting of nice involutions  $w \in G$ . Since they all have the form above, it follows that they generate a subgroup  $\Gamma$  of  $G$ , and thus  $G = \Gamma \ltimes G^0$ , where  $\Gamma$  is a subgroup of  $A_2$ . Note that  $\Gamma$  can only be  $A_1$  or  $A_2$ , since it is non-trivial and contains an element of order 2.  $\square$

Both cases described in Lemma 2 do occur: the first one for the rank 4 real Grassmannian and the second one for the rank 4 exceptional quaternion-Kähler symmetric spaces. We will see below that no further examples exist.

**3.6. Symmetries of  $X$ .** The key point is that  $\rho^0 : \mathrm{Sp}(1)^3 \rightarrow \mathrm{O}(V)$  is a representation of quaternionic type. The centralizer  $\mathrm{Sp}(1)'$  of  $G^0$  in  $\mathrm{O}(V)$  also centralizes  $\Gamma$  and acts thus on  $X = V/G = W/H$ :

$$\begin{array}{c} W \\ \downarrow \\ X = W/H = \otimes^3 \mathbb{C}^2 / \Gamma \cdot \mathrm{Sp}(1)^3 \curvearrowright \mathrm{Sp}(1)' \\ \downarrow \\ Y = \otimes^2 \mathbb{R}^4 / \Gamma \cdot \mathrm{SO}(4)^2 \end{array}$$

Denote the composite map  $W \rightarrow Y$  by  $\pi$ . Since  $(\mathrm{SO}(4) \times \mathrm{SO}(4), \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^4)$  is a polar representation (indeed the isotropy representation of the rank 4 real Grassmannian manifold),  $Y$  is a flat Riemannian orbifold (of dimension 4) and hence the components of the level sets of  $\pi$  yield an isoparametric foliation  $\mathcal{F}$  by full irreducible submanifolds (compare [GL15, §2]). The codimension of  $\mathcal{F}$  is 4, so by a theorem of Thorbergsson [Tho91],  $\mathcal{F}$  is homogeneous, namely, the maximal connected subgroup  $\hat{H}$  of  $\mathrm{O}(V)$  that preserves the leaves of  $\mathcal{F}$  acts transitively on them. By definition,  $\hat{H}$  is closed, acts polarly, and contains  $H$ . In particular, it acts irreducibly on  $W$ . It follows that  $\hat{\tau} : \hat{H} \rightarrow \mathrm{O}(W)$  is the isotropy representation of an irreducible symmetric space, of rank 4.

The geometry of  $Y$  can be understood. The Coxeter group of the polar representation  $(\mathrm{SO}(4) \times \mathrm{SO}(4), \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^4)$  is  $D_4$ , so  $Y = \mathbb{R}^4 / \Gamma'$  where  $\Gamma'$  is a finite extension of  $D_4$  by  $\Gamma$ , of order  $2\#(D_4)$  or  $3!\#(D_4)$ , that acts irreducibly on  $\mathbb{R}^4$ . Recall that  $\mathrm{Aut}(D_4) \subset \mathrm{Ad}(F_4)$ , that is, every automorphism of  $D_4$  becomes inner in  $F_4$  (see e.g. [Ada96, Theorem 14.2]). Since the representation of  $\Gamma'$  on  $\mathbb{R}^4$  is irreducible of real type, we deduce that every element of  $\Gamma'$  differs from an element of  $F_4$  by  $\pm 1$ , but  $-1 \in D_4$ . This proves that  $\Gamma'$  is a subgroup of  $F_4$ . Since  $F_4 = D_4 \ltimes A_2$  and  $B_4 = D_4 \ltimes A_1$ , we deduce that  $\Gamma' = B_4$  or  $F_4$  according to whether  $\Gamma = A_1$  or  $A_2$ .

**3.7. End of the proof.** We refer to the classification of isotropy representations of symmetric spaces [Wol84, ch. 8] to list the possibilities for  $\hat{\tau}$ . If  $\Gamma' = B_4$ , then  $\hat{\tau} : \hat{H} \rightarrow \mathrm{O}(W)$  is one of:

$$\begin{array}{ll} (\mathrm{SO}(n) \times \mathrm{SO}(4), \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^4) & (n \geq 5), \quad (\mathrm{U}(8), \Lambda^2 \mathbb{C}^8), \\ (\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(4)), \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^4) & (n \geq 4), \quad (\mathrm{U}(9), \Lambda^2 \mathbb{C}^9), \\ (\mathrm{Sp}(n) \times \mathrm{Sp}(4), \mathbb{H}^n \otimes_{\mathbb{H}} \mathbb{H}^4) & (n \geq 4), \quad (\mathrm{U}(4), \mathrm{S}^2 \mathbb{C}^4). \end{array}$$

If  $\Gamma' = F_4$ , then  $\hat{\tau}$  is associated to one of the exceptional quaternion-Kähler symmetric spaces of rank 4, namely, the last four cases in the table of the Theorem. In each case, we look for closed subgroups  $H$  of  $\hat{H}$  that act irreducibly with cohomogeneity 7 on  $W$ . A straightforward, not very long calculation using the lists of maximal connected closed subgroups of compact Lie groups compiled by Dynkin ([Dyn52]; see also [GP05]) reduces the possibilities for  $H$  to those listed in the Theorem plus two extra candidates, namely,  $\mathrm{Spin}(7) \times \mathrm{U}(2) \subset \mathrm{SO}(8) \times \mathrm{SO}(4)$  and  $\mathrm{SU}(4) \times \mathrm{Sp}(2) \subset \mathrm{S}(\mathrm{U}(4) \times \mathrm{U}(4))$ , both corresponding to the case  $\Gamma' = B_4$ . In the sequel we rule out those representations by showing that they cannot be quotient-equivalent to  $(\mathbb{Z}_2 \cdot \mathrm{Sp}(1)^3, \otimes^3 \mathbb{C}) = (\mathrm{O}(4) \times \mathrm{Sp}(1), \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^4)$ .

Owing to the discussion in Subsections 3.3 and 3.4, the boundary of the orbit space  $X$  of  $(\mathrm{O}(4) \times \mathrm{Sp}(1), \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^3)$  originates from the fixed point set of a nice



involution  $w \in (\mathrm{O}(4) \setminus \mathrm{SO}(4)) \times \{1\}$ , which we can take to be  $w = \mathrm{diag}(-1, 1, 1, 1)$ . It follows that  $\partial X$  is also given as the orbit space of a representation (and we say that  $X$  has *linear boundary*), namely,  $(\mathrm{O}(3) \times \mathrm{Sp}(1), \mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4)$ . It has already been remarked in Subsection 3.2 that this representation has empty boundary. We deduce that  $X$  contains no strata of codimension 2 along its boundary. Another remark that will be useful below is that the slice representations at non-zero points of  $(\mathrm{O}(3) \times \mathrm{Sp}(1), \mathbb{R}^3 \otimes_{\mathbb{R}} \mathbb{R}^4)$  are all infinitesimally polar. Recall that a representation of a compact Lie group is called *infinitesimally polar* if the slice representations at non-zero points are all polar [GL16]; in the case at hand, this property follows because the isotropy subgroups of  $\mathrm{O}(3) \times \mathrm{Sp}(1)$  can be at most 1-dimensional. Note that infinitesimal polarity for a representation can be detected metrically as it is equivalent to having the orbit space isometric to a Riemannian orbifold [LT10].

It follows easily from [Sch80, §13] that the orbit space of  $(\mathrm{SU}(4) \times \mathrm{Sp}(2), \mathbb{C}^4 \otimes_{\mathbb{C}} \mathbb{H}^2)$  admits no  $S^3$ -boundary components (and it admits no  $\mathbb{Z}_2$ -boundary components, as the underlying group is connected), but we can find two  $S^1$ -boundary components that meet at a codimension 2 strata, namely, the projections of the fixed point sets of the circle subgroups

$$(\mathrm{diag}(e^{i\theta}, e^{i\theta}, e^{-i\theta}, e^{-i\theta}), (e^{-i\theta}, e^{-i\theta}, e^{i\theta}, e^{i\theta}))$$

and

$$(\mathrm{diag}(e^{i\theta}, 1, e^{-i\theta}, 1), (e^{-i\theta}, 1, e^{i\theta}, 1)),$$

as is readily checked. Hence this orbit space cannot be isometric to  $X$ .

Finally, we turn to  $(\mathrm{Spin}(7) \times \mathrm{U}(2), \mathbb{R}^8 \otimes_{\mathbb{R}} \mathbb{C}^2)$  and its orbit space, which we denote by  $Z$ . The slice representation at a point  $p$  given by a pure tensor, say restricted to the identity component, is given by  $(\mathrm{G}_2 \times \mathrm{SO}(2), \mathbb{R}^7 \otimes_{\mathbb{R}} \mathbb{R}^2 \oplus \mathbb{R}^7)$  which, in view of the classification in [GL16], is not infinitesimally polar. On the other hand, the point  $p$  projects to a  $S^3$ -boundary component of  $Z$  given by the projection of the fixed point set of a  $\mathrm{SU}(2)$ -subgroup of  $\mathrm{SU}(3) \subset \mathrm{G}_2 \times \{1\} \subset \mathrm{G}_2 \times \mathrm{SO}(2)$ . Hence  $Z$  cannot be isometric to  $X$ .

That  $\rho$  is a minimal reduction of  $\tau$  follows from the last assertion in Subsection 3.3 and this finishes the proof of the Theorem.

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